

# Number of Walks and Degree Powers in a Graph

M.A. Fiol, E. Garriga

Universitat Politècnica de Catalunya, BarcelonaTech  
 Departament de Matemàtica Aplicada IV  
 Barcelona, Catalonia  
 (e-mails: {fiol, egarriga}@ma4.upc.edu)

## Abstract

This note deals with the relationship between the total number of  $k$ -walks in a graph, and the sum of the  $k$ -th powers of its vertex degrees. In particular, it is shown that the number of all  $k$ -walks is upper bounded by the sum of the  $k$ -th powers of the degrees.

Let  $G = (V, E)$  be a connected graph on  $n$  vertices,  $V = \{1, 2, \dots, n\}$ , with adjacency matrix  $\mathbf{A}$ . For any integer  $k \geq 1$ , let  $a_{ij}^{(k)}$  denote the  $(i, j)$  entry of the power matrix  $\mathbf{A}^k$ . Let  $\mathbf{D}$  be the diagonal matrix with elements  $(\mathbf{D})_{ii} = d_i$  (the degree of vertex  $i$ ). Here we study the relationship between the sum of all walks of length  $k$  in  $G$  and the sum of the  $k$ -th powers of its degrees. As a main result, and answering in the affirmative a conjecture of Marc Noy [8], we will show that

$$\sum_{i,j} a_{ij}^{(k)} \leq \sum_i d_i^k, \quad (1)$$

with equality if and only if  $G$  is regular or  $k \leq 2$ . In the case  $k = 3$  we also provide an exact value of the difference between the above sums in (1). In other line of research, some upper bounds for  $\sum_i d_i^k$  have been given by several authors. See, for instance, [3, 6, 7, 10] (for general graphs) and [2, 5] (for graphs not containing a prescribed subgraph).

Let us first begin with the small values of  $k$ . The case  $k = 0$  is trivial since the number of walks of length 0 equals the number of vertices. Similarly, if  $k = 1$ , the sum  $\sum_{i,j} a_{ij}$  is just the sum of the degrees  $d_1 + d_2 + \dots + d_n$ . If  $k = 2$ , we can use that  $\mathbf{A}\mathbf{j} = \mathbf{D}\mathbf{j}$  (where  $\mathbf{j}$  is the all-1 vector) and the symmetry of the involved matrices to obtain:

$$\sum_{i,j} a_{ij}^{(2)} = \langle \mathbf{j}, \mathbf{A}^2 \mathbf{j} \rangle = \langle \mathbf{A}\mathbf{j}, \mathbf{A}\mathbf{j} \rangle = \|\mathbf{A}\mathbf{j}\|^2 = \|\mathbf{D}\mathbf{j}\|^2 = d_1^2 + d_2^2 + \dots + d_n^2.$$

Assume now that  $G$  is regular of degree, say,  $d$ . Then,  $\mathbf{j}$  is the positive eigenvector corresponding to the eigenvalue  $d$  and we get

$$\sum_{i,j} a_{ij}^{(k)} = \langle \mathbf{j}, \mathbf{A}^k \mathbf{j} \rangle = \langle \mathbf{j}, d^k \mathbf{j} \rangle = d^k \|\mathbf{j}\|^2 = nd^k.$$

A similar reasoning shows that, for a general (non-regular) graph, the inequality in (1) always holds for  $k$  large enough. Indeed, let  $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_n)^\top$  be the positive eigenvector of  $G$ , normalized in such a way that  $\min_{i \in V} \nu_i = 1$ . Let  $\lambda$  be its corresponding (positive) eigenvalue, which is known to be smaller than the maximum degree  $\Delta$  of  $G$  (see [1, 4]). Then,

$$\sum_{i,j} a_{ij}^{(k)} \leq \langle \boldsymbol{\nu}, \mathbf{A}^k \boldsymbol{\nu} \rangle = \langle \boldsymbol{\nu}, \lambda^k \boldsymbol{\nu} \rangle = \|\boldsymbol{\nu}\|^2 \lambda^k$$

which, for  $k$  large enough, is smaller than the single  $k$ -power  $\Delta^k$ .

To deal with the case  $k = 3$ , it is more convenient to work with the Laplacian matrix of  $G$ ; that is,  $\mathbf{L} := \mathbf{D} - \mathbf{A}$ . Then, recall that, given any real function defined on  $V$ ,  $f : V \rightarrow \mathbb{R}$ , and with  $\mathbf{f}$  being the (column) vector with components the values of  $f$  on  $V$ , we have

$$\langle \mathbf{f}, \mathbf{L} \mathbf{f} \rangle = \sum_{i \sim j} (f(i) - f(j))^2,$$

where the sum is extended over all edges of  $G$  (see, for instance, [1]). We are interested in the case when the above function is just the degree of the corresponding vertex:  $f(i) = d_i$ . In this case, we denote by  $\boldsymbol{\delta}$  its corresponding vector. Then, the difference between the two sums in (1) is just

$$\sum_i d_i^3 - \sum_{i,j} a_{ij}^{(3)} = \langle \boldsymbol{\delta}, \mathbf{D} \boldsymbol{\delta} \rangle - \langle \boldsymbol{\delta}, \mathbf{A} \boldsymbol{\delta} \rangle = \langle \boldsymbol{\delta}, \mathbf{L} \boldsymbol{\delta} \rangle = \sum_{i \sim j} (d_i - d_j)^2 \geq 0.$$

To prove the inequality in the general case, note first that, for any two positive numbers  $a, b$  with, say,  $a \geq b$ ,

$$a^r b + ab^r = a^{r+1} + b^{r+1} - (a^r - b^r)(a - b) \leq a^{r+1} + b^{r+1}$$

with equality if and only if  $a = b$ . (The same conclusion is reached when we apply Hölder inequality [8] to the vectors  $(a, b)$  and  $(b^r, a^r)$  with norms  $L^{r+1}$  for the first vector and dual norm  $L^{(r+1)/r}$  for the second one.)

Also, notice that all walks of a given length, say  $k \geq 1$ , can be obtained by considering, for any vertex  $i$ , all  $(i, j)$ -walks of length  $k - 1$  “extended” by each of the  $d_i$  edges incident to  $i$ . Thus,  $\sum_{i,j} a_{ij}^{(k)} = \sum_{i,j} d_i a_{ij}^{(k-1)}$  or, equivalently,

$$\langle \mathbf{j}, \mathbf{A}^k \mathbf{j} \rangle = \langle \mathbf{j}, \mathbf{A}^{k-1} \mathbf{D} \mathbf{j} \rangle = \langle \mathbf{D} \mathbf{j}, \mathbf{A}^{k-1} \mathbf{j} \rangle.$$

Keeping all this in mind, we are now ready to prove (1). Indeed, assuming that  $k \geq 3$ , we have:

$$\begin{aligned}
\sum_{i,j} a_{ij}^{(k)} &= \sum_{i,j} d_i a_{ij}^{(k-2)} d_j = \sum_i a_{ii}^{(k-2)} d_i^2 + \sum_{i < j} 2a_{ij}^{(k-2)} d_i d_j \\
&\leq \sum_i a_{ii}^{(k-2)} d_i^2 + \sum_{i < j} a_{ij}^{(k-2)} (d_i^2 + d_j^2) \\
&= \sum_{i,j} a_{ij}^{(k-2)} d_j^2 \\
&= \sum_{i,j} d_i a_{ij}^{(k-3)} d_j^2 = \sum_i a_{ii}^{(k-3)} d_i^3 + \sum_{i < j} a_{ij}^{(k-3)} (d_i d_j^2 + d_i^2 d_j) \\
&\leq \sum_i a_{ii}^{(k-3)} d_i^3 + \sum_{i < j} a_{ij}^{(k-3)} (d_i^3 + d_j^3) \\
&= \sum_{i,j} a_{ij}^{(k-3)} d_j^3 \leq \cdots \leq \sum_{i,j} a_{ij} d_j^{k-1} = \sum_j d_j^k.
\end{aligned}$$

Moreover, notice that all the above inequalities become equalities if and only if  $G$  is regular, as claimed.

As commented by one of the referees, the above prove suggests a wider result: If  $\mathbf{A}$  is a real symmetric  $n \times n$  matrix,  $\mathbf{D} = \text{diag}(d_1, d_2, \dots, d_n)$ , where  $d_i = \sum_j |a_{ij}|$ , and  $\mathbf{J}$  is the all-1 matrix, then

$$\text{tr}(\mathbf{A}^k \mathbf{J}) \leq \text{tr}(\mathbf{D}^k \mathbf{J}) = \sum_i d_i^k. \quad (2)$$

In other words, an extremal (not characteristic) property of the positive diagonal matrices is exhibited.

## Acknowledgments

Research supported by the Ministerio de Educación y Ciencia, Spain, and the European Regional Development Fund under project MTM2005-08990-C02-01 and by the Catalan Research Council under project 2005SGR00256.

The authors thank one of the referees for the comment about the use of Hölder inequality and the result in (2).

## References

- [1] N. Biggs, *Algebraic Graph Theory*, second ed., Cambridge University Press, Cambridge, 1993.

- [2] B. Bollobás and V. Nikiforov, Degree powers in graphs with forbidden subgraphs, *Electron. J. Combin.* 11 (2004), Research paper R42.
- [3] Sebastian M. Cioabă, Sums of powers of the degrees of a graph, *Discrete Math.* 306 (2006) 1959–1964.
- [4] D.M. Cvetković, Graphs and their spectra, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No. 354–356 (1971), 1–50.
- [5] Y. Caro and R. Yuster, A Turán type problem concerning the powers of the degrees of a graph, *Electron. J. Combin.* 7 (2000), Research paper R47.
- [6] K. Ch. Das, Maximizing the sum of the squares of the degrees in a graph, *Discrete Math.* 285 (2004), 57–66.
- [7] D. de Caen, An upper bound on the sum of the squares of the degrees in a graph, *Discrete Math.* 185 (1998), 245–248.
- [8] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.
- [9] Marc Noy, personal communication (2005).
- [10] U.N. Peled, R. Petreschi and A. Sterbini,  $(n, e)$ -graphs with maximum sum of squares of degrees, *J. Graph Theory* 31 (1999), 283–295.